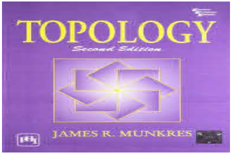
Topology



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Continuous function

Definition 1.

Let X and Y be topological spaces.

A function $f : X \to Y$ is continuous if for each open subset V of Y,

the set $f^{-1}(V)$ is open in X.

Lemma A.

Let $f : X \to Y$.

Let $\mathcal{B}$ be a basis for the topology on Y and

let $\mathcal{S}$ be a subbasis for the topology on Y.

(1) f is continuous if $f^{-1}(B)$ is open in X for each $B \in \mathcal{B}$.

(2) f is continuous if $f^{-1}(S)$ is open in X for each $S \in \mathcal{S}$.

Theorem 1. Let X and Y be topological spaces.

Let $f : X \to Y$. Then the following are equivalent:

(1) f is continuous.

(2) For every subset A of X, one has $f(\overline{A}) \subset \overline{f(A)}$.

(3) For every closed subset B of Y, the set $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a

neighborhood U of x such that $f(U) \subset V$.

Definition 2.

Let X and Y be topological spaces.

Let $f : X \to Y$ be a bijection (one to one and onto).

If both f and $f^{-1} : Y \to X$ are continuous.

Then f is a homeomorphism.

Definition 3.

Let $f : X \to Y$ be an injective (one to one) continuous map.

Let $Z = f(X)$ (so that f is onto Z) be considered a subspace of Y.

Let $f' : X \to Z$ be the restriction of f to Z (so f' is a bijection).

If f' is a homeomorphism of X with Z.

Then $f : X \to Y$ is a *topological imbedding* of X in Y.

Theorem 2.

Let X, Y, and Z be topological spaces.

(a) (Constant Function)

If $f : X \to Y$ maps all of X into a single point $y_0 \in Y$,

then f is continuous.

(b) (Inclusion)

If A is a subspace of X, then the inclusion function $j : A \to X$

is continuous.

(c) (Composites)

If $f : X \to Y$ and $f : X \to Y$ are continuous, then the map

$g \circ f : X \to Y$ is continuous.

(d) (Restricting the Domain)

If $f : X \to Y$ is continuous and if A is a subspace of X, then the

restricted function $f \mid A : A \to Y$ is continuous.

(e) (Restricting or Expanding the Range) let $f : X \to Y$.

If X is a subspace of Y containing the image set $f(X)$, then the

function $g : X \to Z$ obtained by restricting the range of f is continuous.

If Z is a space having Y as a subspace, then the functions

$h : X \to Z$ obtained by expanding the range of f is continuous.

(f) (Local Formulation of Continuity) The map $f : X \to Y$ is continuous.

If X can be written as the union of open sets U_α such that

$f \mid U_\alpha$ is continuous for each α.

Theorem 3.

Let $X = A \cup B$ where A and B are closed sets in X.

Let $f : A \to Y$ and $g : B \to Y$ be continuous.

If $f(x) = g(x)$ for all $x \in A \cap B$, then f and g combine (or paste) to give

a continuous function $h : X \to Y$ defined by setting

$h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

Theorem 4.

Let $f : A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$

where $f_1 : A \to X$ and $f_2 : A \to Y$.

Then f is continuous if and only if the functions f_1 and f_2 are continuous.

Definition 3.

Let J be an index set.

Given a set X, define a J-tuple of elements of X to be

a function $x : J \to X$.

If α is an element of J, we denote the value of x at α by x_α

rather than $x(\alpha)$ called the αth coordinate of x.

We often denote x as $(x_\alpha)_{\alpha \in J}$ and denote the set of all

J-tuples of elements of X as X^J.

Definition 4.

A metric on a set X is a function $d : X \times X \to \mathbb{R}$ having

the following properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) (The Triangle Inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

Definition 5.

If d is a metric on X then the collection of all ϵ- balls $B_d(x, \epsilon)$ for $x \in X$

(where $B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$ and $\epsilon > 0$)

is a basis for a topology on X, called the

metric topology induced by d.

Lemma A.

Let $B_d(x, \epsilon)$ be a ϵ- ball in a topological space with

the metric topology and metric d.

Let $y \in B_d(x, \epsilon)$.

Then there is $\delta > 0$ such that $B_d(y, \delta) \subset B_d(x, \epsilon)$.

Lemma B.

A set U is open in the metric topology induced by metric d

if and only if for each $y \in U$ there is a $\delta > 0$ such that

$B_d(y, \delta) \subset U$.

Definition 6.

Let X be a topological space.

X is said to be metrizable if there exists a metric d on a set X

that induces the topology of X.

A metric space is a metrizable space X with a specific metric d

that gives the topology of X.

Definition 7.

Let X be a metric space with metric d.

A subset A of X is bounded if there is some number M such that

$d(a_1, a_2) \leq M$ for every pair $a_1, a_2 \in A$.

If A is bounded and nonempty, then the diameter of A is

$\text{diam(A)} = \sup \{ d(a_1, a_2) \,|\, a_1, a_2 \in A \}$.

Theorem 5.

Let X be a metric space with metric d.

Define $\overline{d} : X \times X \to \mathbb{R}$ by

$$\overline{d}(x, y) = min\{d(x, y), 1\}.$$

Then $\overline{d}$ is a metric that induces the same topology as d.

Definition 8.

Given $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, define the norm of x as

$$|| \, x \, || = \, ({x_1}^2 + {x_2}^2 + \ldots + {x_n}^2)^{1/2}.$$

Define the Euclidean metric on $\mathbb{R}^n$ as

$$d(x, y) \, = || \, x - y \, || = \, ((x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2)^{1/2}.$$

Define the square metric ρ as

$$\rho(x, y) \, = \, max \, \{ \, |x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n| \, \}.$$

Lemma 2.

Let d and d' be two metrics on the set X.

Let τ and τ' be the topologies they induce, respectively.

Then τ' is finer than τ if and only if for each $x \in X$ and each $\epsilon > 0$,

there exists a $\delta > 0$ such that $B'_d(x, \delta) \subset B_d(x, \epsilon)$.

Theorem 3.

The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and

the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Theorem 4.

The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology.

These three topologies are all different if J is infinite.

Theorem 5.

Let $d(a, b) = min \{|a - b|, 1\}$ be the standard bounded metric on $\mathbb{R}$.

If x and y are two points in $\mathbb{R}^\omega = \mathbb{R}^n$, define

$$D(x, y) = sup_{i \in \mathbb{N}} \left\{ \frac{\overline{d}(x_i, y_i)}{i} \right\}$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$.

That is, $\mathbb{R}^\omega$ under the product topology is metrizable.

Theorem 1.

Let $f : X \to Y$. Let X and Y be metrizable with metrics d_X and d_Y, respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon.$$

Lemma 2.

Let X be a topological space. Let $A \subset X$.

If there is a sequence of points of A converging to x, then $x \in \bar{A}$.

If X is metrizable and $x \in \bar{A}$ then there is a sequence $\{x_n\} \subset A$

such that $\{x_n\} \to x$.

Theorem 3.

Let $f : X \to Y$. If f is continuous then for every convergent sequence $\{x_n\} \to x$ in X, the sequence $\{f(x_n)\} \to f(x)$ in Y.

If X is metrizable and for any sequence $\{x_n\} \to x$ in X, we have $\{f(x_n)\} \to f(x)$ in Y then f is continuous.

Definition 9.

A topological space X is said to have a **countable basis** at the point x if there is a countable collection $\{U_n\}_{n\in\mathbb{N}}$ of neighborhoods of x such that any neighborhood U of x contains at least one of the sets U_n.

A space that has a countable basis at each of its points is said to satisfy the **first countability axiom**.

Lemma 4.

The addition, subtraction, and multiplication operations are continuous from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and the quotient operation is a continuous function from $\mathbb{R} \times (\mathbb{R} - \{0\})$ into $\mathbb{R}$.

Theorem 5.

If X is a topological space and if $f, g : X \to \mathbb{R}$ are continuous,

then $f + g, f - g$, and $f \circ g$ are continuous.

If $g(x) \neq 0$ for all $x \in X$ then f/g is continuous.

Definition 10.

Let $f : X \to Y$ be a sequence of functions from set X to metric space Y.

let d be the metric for Y.

The sequence of functions $\{f_n\}$ converges uniformly to the function

$f : X \to Y$ if given $\epsilon > 0$ there is $n \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) < \epsilon \text{ for all } n > N \text{ and for all } x \in X.$$

Theorem 6.

Let $f_n : X \to Y$ be a sequence of continuous functions from

the topological space X to the metric space Y.

If $\{f_n\}$ converges uniformly to f, then f is continuous.